CONSISTENT SOLUTION OF MARKOV'S PROBLEM ABOUT ALGEBRAIC SETS

OL'GA V. SIPACHEVA

Abstract. It is proved that the continuum hypothesis implies the existence of a group M containing a nonalgebraic unconditionally closed set, i.e., a set which is closed in any Hausdorff group topology on M but is not an intersection of finite unions of solution sets of equations in M.

Definition 1 (Markov [1]). A subset A of a group G is said to be *unconditionally closed* in G if it is closed in any Hausdorff group topology on G.

Clearly, all solution sets of equations in G, as well as their finite unions and arbitrary intersections, are unconditionally closed. Such sets are called algebraic. The precise definition is as follows.

Definition 2 (Markov [1]). A subset A of a group G with identity element 1 is said to be *elementary algebraic* in G if there exists a word $w = w(x)$ in the alphabet $G \cup \{x^{\pm 1}\}$ (x is a variable) such that

$$A = \{x \in G : w(x) = 1\}.$$

Finite unions of elementary algebraic sets are called *additively algebraic* sets. An arbitrary intersection of additively algebraic sets is said to be *algebraic*. Thus, the algebraic sets in G are the solution sets of arbitrary conjunctions of finite disjunctions of equations.

In his 1945 paper [1], A. A. Markov showed that any algebraic set is unconditionally closed and posed the problem of whether the converse is true. In [2] (see also [3]), he solved this problem for countable groups by proving that any unconditionally closed set in a countable group is algebraic. The answer is also positive for subgroups of direct products of countable groups [4].

Markov's problem is closely related to the topologizability of groups. Recall that a group is said to be *topologizable* if it admits a nondiscrete Hausdorff group topology. Groups that are not topologizable are called *nontopologizable*. The problem of the existence of a nontopologizable group was posed by Markov in the same 1945 paper [1]; it was solved under CH by Shelah in 1976 (published in 1980 [5]). The first ZFC

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example was given by Hesse in 1979 [6]; a year later, Ol'shanskii constructed a countable nontopologizable group in ZFC [7]. More recent results can be found in [8].

In this paper, we prove the following theorem.

Theorem. *Under CH, there exists a group containing a nonalgebraic unconditionally closed set.*

Proof. Such a group is the nontopologizable group M constructed by Shelah [5]. It has many remarkable properties. What we need is

$$M = \bigcup_{\alpha\in\omega_1} M_\alpha,$$

where each M_α is a countable subgroup of M_β for any $\beta \ni \alpha$ and all of the M_α (except possibly M_1) are increasing unions of topologizable subgroups. The following general observation shows that this is sufficient for M to have a nonalgebraic unconditionally closed subset.

Lemma 1. *If G is a nontopologizable group and any finite subset of G is contained in a topologizable subgroup of G, then $G \setminus \{1\}$ is a nonalgebraic unconditionally closed subset of G.*

Proof. Since G admits no nondiscrete Hausdorff group topology, the set $A = G\setminus\{1\}$ is unconditionally closed in G. Suppose that it is algebraic. Then $A = \bigcap_{\gamma\in\Gamma} A_\gamma$, where Γ is an arbitrary index set and each A_γ is an additively algebraic set in G. All of the sets A_γ must contain $G \setminus \{1\}$; hence each of them must coincide with G or $G \setminus \{1\} = A$. Clearly, some of these sets does not contain 1; thus, $A = A_\gamma$ for some γ. This means that $A = \bigcup_{i\le k} A_i$, where $k \in \omega$ and each A_i is an elementary algebraic set. This means that there exist words $w_1(x), \dots, w_k(x)$ in the alphabet $G \cup \{x^{\pm1}\}$ such that

$$A_i = \{x \in G : w_i(x) = 1\}$$

for $i \le k$. Since the number of letters in each word is finite, we can find a topologizable subgroup $H \subset G$ such that all of the $w_i(x)$ are words in the alphabet $H \cup \{x^{\pm1}\}$. Thus, the $A_i \cap H$ are elementary algebraic sets in H, and $A\cap H = H\setminus\{1\}$ is an algebraic (and, therefore, unconditionally closed) set in H, which contradicts the topologizability of H. □

Remark 1. Combining Lemma 1 with the theorem of Markov about unconditionally closed sets in countable groups, we see that any countable group which is an increasing union of topologizable subgroups is topologizable. In particular, all of the groups M_α, except possibly M_1, are topologizable, and the group M is uncountable.

This essentially completes the proof of the theorem. It only remains to verify that M has sufficiently many topologizable subgroups.[1] This requires knowledge of the structure of the groups M_α. Below, we reproduce (or, to be more precise, reconstruct) the part of Shelah's proof containing the construction of these groups, which is far from being overloaded with details, in contrast to misprints and lacunae. The description of Shelah's group suggested below slightly differs from that given in [5], but the essence is the same. The proof uses the notions of a malnormal subgroup and good fellows over a subgroup. Recall that a subgroup H in a group G is said to be *malnormal* if $g^{-1}Hg \cap H = \{1\}$ for any $g \in G \setminus H$. Shelah calls two elements x and y of a group G *good fellows*[2] over a subgroup $H \subset G$ if $x, y \in G \setminus H$ and the double cosets $Hx^\varepsilon H$ and $Hy^\delta H$ are disjoint for $\varepsilon, \delta = \pm 1$, i.e., $x \notin Hy^{\pm 1}H$. Other algebraic notions, constructions, and facts used in the proof are collected in the appendix; the very basic definitions can be found in [9].

The groups M_α are constructed by induction as follows. Let

$$\{S_\gamma : \gamma \in \omega_1\}$$

be the family of all infinite countable subsets of $\omega \times \omega_1$ enumerated in such a way that $S_0 = \omega \times \{0\}$ (recall that we have assumed $\mathfrak{c} = \omega_1$). Let M_0 be the trivial group. For M_1 we take an arbitrary non–finitely generated countable group and identify it (as a set) with $\omega \times \{0\}$. Suppose that $\alpha \in \omega_1$ and M_α is already constructed. We identify it with $\omega \times \alpha$ (each ordinal is considered as the set of all smaller ordinals). Let us construct $M_{\alpha+1}$.

Consider the set

$$T_\alpha = \omega^2 \times \alpha \times M_\alpha$$

of all triples $((i,j),\gamma,h)$, where $i,j \in \omega$, $\gamma \in \alpha$, and $h \in M_\alpha$. This set is countable. Let us enumerate it:

$$T_\alpha = \{((i_n, j_n), \gamma_n, h_n) : n \in \omega\};$$

we require that $i_n \le n$ for any n. (Certainly, each of i, j, γ, and h occurs in T_α infinitely many times.) First, we construct increasing sequences of countable groups H_n^α and L_n^α such that

(i) each H_n^α is a finitely generated subgroup of M_α, and $\bigcup H_n^\alpha = M_\alpha$;

(ii) each H_n^α is a subgroup of L_n^α, each L_n^α is a subgroup of L_{n+1}^α, and $L_n^\alpha \cap M_\alpha = H_n^\alpha$;

[1] It is mentioned in [5] without proof that all countable subgroups of M are topologizable. This is not so unless special care is taken; at least, the group M_0, which is the basis of the inductive construction of M, must be topologizable.

[2] In the definition of good fellows given by Shelah in [5, p. 377], "$G - H$" should read "$H - G$".

(iii) the set $L^\alpha_n \setminus M_\alpha$ is infinite, and its elements are indexed by pairs of integers:

$$L^\alpha_n \setminus M_\alpha = L^\alpha_n \setminus H^\alpha_n = \{a_{(i,j)} : i \le n,\ j \in \omega\};$$

(iv) if $S_{\gamma_n} \subset M_\alpha$ and S_{γ_n} is contained in no finitely generated subgroup of M_α, then

$$h_n \in \big((S_{\gamma_n} \cap H^\alpha_{n+1}) \cup \{a_{(i_n,j_n)}\}\big)^{10000} \subset L^\alpha_{n+1}$$

(this is the usual power of a set in the group L^α_{n+1});

(v) H^α_n is a malnormal subgroup of L^α_n, i.e., $a^{-1}H^\alpha_n a \cap H^\alpha_n = \{1\}$ for any $a \in L^\alpha_n \setminus H^\alpha_n$.

The groups L^α_n and H^α_n are defined by induction. We set $H^\alpha_0 = \{1\}$ and let L^α_0 be an infinite cyclic group having trivial intersection with M_α. We somehow enumerate the elements of $L^\alpha_0 \setminus \{1\}$ by pairs from $\{0\} \times \omega$:

$$L^\alpha_0 \setminus \{1\} = \{a_{(0,j)} : j \in \omega\}.$$

Suppose that H^α_n and L^α_n are constructed and

$$L^\alpha_n \setminus M_\alpha = \{a_{(i,j)} : i \le n,\ j \in \omega\}.$$

Let us construct H^α_{n+1} and L^α_{n+1}. Recall that we have enumerated all infinite countable subsets of $\omega \times \omega_1$ at the very beginning of the construction and that M_α is identified with $\omega \times \alpha$. If the set S_{γ_n} (the γ_n is from the enumeration of the set T_α of triples) is not contained in M_α or is contained in a finitely generated subgroup of M_α, then we set $H^\alpha_{n+1} = \langle H^\alpha_n, h_n\rangle$ (this is the subgroup generated by H^α_n and h_n in M_α; it is finitely generated by the induction hypothesis) and $L^\alpha_{n+1} = L^\alpha_n *_{H^\alpha_n} H^\alpha_{n+1}$ (this is the free product of L^α_n and H^α_{n+1} with amalgamation over H^α_n; see the appendix). Otherwise, i.e., if S_{γ_n} is contained in M_α and is not contained in any finitely generated subgroup of M_α, then there exist $x, y \in S_{\gamma_n} \setminus H^\alpha_n$ such that $x \notin H^\alpha_n y^{\pm 1} H^\alpha_n \cup h_n H^\alpha_n$ in M_α (in particular, x and y are good fellows over H^α_n). The proof is similar to that of Fact 2.2(ii) from [5]: if any element of $S_{\gamma_n} \setminus H^\alpha_n$ would belong to $H^\alpha_n z H^\alpha_n \cup H^\alpha_n z^{-1} H^\alpha_n \cup h_n H^\alpha_n$, where z is an arbitrary element of $S_{\gamma_n} \setminus H^\alpha_n$, then S_{γ_n} would be contained in the set $H^\alpha_n z H^\alpha_n \cup H^\alpha_n z^{-1} H^\alpha_n \cup h_n H^\alpha_n \cup H^\alpha_n$, which is in turn contained in a finitely generated subgroup, because H^α_n is finitely generated (by the induction hypothesis). In this case, we set

$$H^\alpha_{n+1} = \langle H^\alpha_n, x, y, h_n\rangle$$

(this subgroup is finitely generated). Recall that T_α is indexed in such a way that $i_n \le n$, so the element $a_{(i_n,j_n)} \in L_{i_n}$ is already defined, and that H^α_n is malnormal in L^α_n by the induction hypothesis. Moreover, by construction, $h_n^{-1}x \in H^\alpha_{n+1} \setminus H^\alpha_n$. We set $h = h_n^{-1}x$ and consider the

word

$$r_0 = ha_{(i_n,j_n)}ya_{(i_n,j_n)}xa_{(i_n,j_n)}(ya_{(i_n,j_n)})^2xa_{(i_n,j_n)}(ya_{(i_n,j_n)})^3 \\ \dots xa_{(i_n,j_n)}(ya_{(i_n,j_n)})^{80} \in L_n^\alpha *_{H_n^\alpha} H_{n+1}^\alpha.$$

Let N be the normal subgroup generated by this word in $L_n^\alpha *_{H_n^\alpha} H_{n+1}^\alpha$. We set

$$L_{n+1}^\alpha = (L_n^\alpha *_{H_n^\alpha} H_{n+1}^\alpha)/N = \langle L_n^\alpha *_{H_n^\alpha} H_{n+1}^\alpha \mid \\ h_n = xa_{(i_n,j_n)}ya_{(i_n,j_n)}xa_{(i_n,j_n)}(ya_{(i_n,j_n)})^2 \dots xa_{(i_n,j_n)}(ya_{(i_n,j_n)})^{80}\rangle$$

(this is the amalgamated free product of L_n^α and H_{n+1}^α with one defining relation $r_0 = 1$). According to Lemma A.2 and the paragraph after this lemma in the appendix, the groups L_n^α and H_{n+1}^α are naturally embedded in L_{n+1}^α as subgroups, and hence $L_n^\alpha \cap H_{n+1}^\alpha = H_n^\alpha$; moreover, by Lemma A.3 from the appendix, H_{n+1}^α is malnormal in L_{n+1}^α. Let us somehow enumerate the elements of $L_{n+1}^\alpha \setminus (L_n^\alpha \cup M_\alpha)$ by the elements of $\{n+1\} \times \omega$.

The construction of the groups H_n^α and L_n^α is completed. The H_n^α satisfy condition (i) because $h_n \in H_n^\alpha$ for every n and $\{h_n : n \in \omega\} = M_\alpha$ by the definition of T_α. The remaining conditions (ii)–(v) hold by construction (10000 is taken as an upper bound for the length of the word r_0).

We set $M_{\alpha+1} = \bigcup L_n^\alpha$.

Finally, we define $M_\beta = \bigcup_{\alpha\in\beta} M_\alpha$ for limit β and set $M = \bigcup_{\alpha\in\omega_1} M_\alpha$.

We have constructed the required group M. As mentioned, it has many remarkable properties. In particular, each M_α is a malnormal subgroup of M (i.e., $h^{-1}M_\alpha h \cap M_\alpha = \{1\}$ for any $h \in M \setminus M_\alpha$) and $S^{10000} = M$ for any uncountable $S \subset M$ (see Lemma 2 below). This immediately implies that M admits no nondiscrete Hausdorff group topology. Indeed, suppose that such a topology exists. Take an arbitrary neighborhood U of the identity element and consider a neighborhood V for which $V^{10000} \subset U$. If V is countable, then it is contained in some M_α and, since M_α is malnormal in M, $h^{-1}Vh \cap V = \{1\}$ for any $h \in M \setminus M_\alpha$; thus, $\{1\}$ is an open set, which contradicts the nondiscreteness of the topology. Hence V must be uncountable, and $M = V^{10000} \subset U$.

Lemma 2. *Each M_α is a malnormal subgroup of M and $S^{10000} = M$ for any uncountable $S \subset M$.*

The malnormality of M_α in M easily follows from the construction. Indeed, it is sufficient to show that M_α is malnormal in $M_{\alpha+1}$ for each α. If $h \in M_{\alpha+1} \setminus M_\alpha$ and $h^{-1}M_\alpha h \cap M_\alpha \neq \{1\}$, then there exist $k, l, m \in \omega$ and $a, b \in M_\alpha$ such that $h \in L_k^\alpha \setminus M_\alpha$, $a \in H_l^\alpha$, $b \in H_m^\alpha \setminus \{1\}$, and $h^{-1}ah = b$. For $n = \max\{k, l, m\}$, we have $h \in L_n^\alpha \setminus M_\alpha = L_n^\alpha \setminus H_n^\alpha$, $a \in H_n^\alpha$, and $b \in H_n^\alpha \setminus \{1\}$; thus, $h^{-1}H_n^\alpha h \cap H_n^\alpha \neq \{1\}$, which contradicts (v).

Let us prove that $S^{10000} = M$ for any uncountable S. First, note that if $S \subset M$ is uncountable, then there exists a β such that $S \cap M_\beta$ is contained in no finitely generated subgroup of M_β. Indeed, take an increasing sequence of countable ordinals β_k such that $S \cap M_{\beta_0} \neq \varnothing$ and $S \cap M_{\beta_{k+1}} \setminus M_{\beta_k} \neq \varnothing$ for any k. Let $\beta = \sup\{\beta_k\}_{k=0}^\infty$. By definition, $M_\beta = \bigcup_{\lambda \in \beta} M_\alpha$. Any subgroup of M_β generated by finitely many elements $g_1, \dots, g_n$ is contained in M_α for some $\alpha < \beta$ and, therefore, in M_{β_k} for some k. Thus, S is not contained in any finitely generated subgroup of M_β. According to Fact 2.8 in [5], S is not contained in any finitely generated subgroup of M_α for any $\alpha \ni \beta$. We have $S \cap M_\beta = S_\gamma$ for some γ. Take any $h \in M$ (then $h \in M_\delta$ for some δ). Since S is uncountable, there exists an $\alpha \ni \max\{\beta, \gamma, \delta\}$ such that $S \cap (M_{\alpha+1} \setminus M_\alpha) \neq \varnothing$. Let $a \in S \cap (M_{\alpha+1} \setminus M_\alpha)$. Then $a \in L_k^\alpha \setminus M_\alpha$ for some k and, by (iii), $a = a_{(i,j)}$ for some $(i,j) \in \omega^2$ $(i \leq k)$. We have $((i,j), \gamma, h) \in T_\alpha$, i.e., $((i,j), \gamma, h) = ((i_n, j_n), \gamma_n, h_n)$ for some n; in particular, $a_{(i,j)} = a_{(i_n,j_n)}$, $S_\gamma = S_{\gamma_n}$, and $h = h_n$. The set $S_{\gamma_n} = S_\gamma = S \cap M_\beta$ is contained in $M_\alpha \supset M_\beta$ but not in a finitely generated subgroup of M_α; hence, by the construction of L_{n+1}^α, there exist $x, y \in S_{\gamma_n} \subset S$ such that $h_n = xa_{(i_n,j_n)}ya_{(i_n,j_n)}xa_{(i_n,j_n)}(ya_{(i_n,j_n)})^2 \dots xa_{(i_n,j_n)}(ya_{(i_n,j_n)})^{80}$ in L_{n+1}^α (and in M). Thus, $h = h_n$ is a product of length less than 10000 of elements of S. $\square$

It remains to prove that M has sufficiently many topologizable subgroups. It suffices to show that, for any $\alpha \in \omega_1 \setminus \{0\}$ and $k \in \omega$, there exists an $n \geq k$ such that the group L_n^α is topologizable. This is implied by Lemma A.4 from the appendix. Indeed, note that, for any $\alpha \in \omega_1 \setminus \{0\}$ and $k \in \omega$, there exists an $n \geq k$ such that the group H_{n+1}^α contains a pair of goods fellows over H_n^α, because, according to Fact 2.8 in [5], any set S not contained in a finitely generated subgroup of some M_α is not contained in any finitely generated subgroup of M_β for $\beta > \alpha$. The group M_1 is not finitely generated; therefore, it is not contained in a finitely generated subgroup of any of the groups M_α. On the other hand, $M_1 = \omega \times \{0\} = S_0$. Each ordinal $\gamma \in \alpha$ occurs in infinitely many triples from T_α; take a triple containing $\gamma = 0$ and having number $n(k) \geq k$ in the enumeration of T_α. By construction, the group $H_{n(k)+1}^\alpha$ is generated by $H_{n(k)}^\alpha$, some element t of M_α, and a pair of goods fellows x, y over $H_{n(k)}^\alpha$, for which $t^{-1}x = h \in H_{n(k)+1}^\alpha \setminus H_{n(k)}^\alpha$; moreover, there exists an $a \in L_{n(k)}^\alpha$ such that $L_{n(k)+1}^\alpha = \langle L_{n(k)}^\alpha *_{H_{n(k)}^\alpha} H_{n(k)+1}^\alpha \mid r_0 = 1 \rangle$, where r_0 is the same word as in Lemma A.4. To obtain the required assertion, it remains to recall that $H_{n(k)}^\alpha$ is malnormal in $L_{n(k)}^\alpha$ by (v) and take $L = L_{n(k)}^\alpha$, $K = H_{n(k)+1}^\alpha$, and $H = H_{n(k)}^\alpha$ in Lemma A.4.

The topologizability of infinitely many groups L_n^α for every nonzero α implies that any finite subset of M is contained in a topologizable subgroup. Indeed, any such subset F is contained in $M_{\alpha+1}$ for some α. On the other hand, $M_{\alpha+1}$ is the union of the increasing sequence

of the groups L^α_n; hence F is contained in L^α_k for some $k \in \omega$. Any topologizable group $L^\alpha_{n(k)}$ with $n(k) \ge k$ contains F.

Since M_1 is an arbitrary non–finitely generated countable group, any at most countable group can be embedded as a subgroup in a group having the same properties as M. We obtain the following corollary.

Corollary. *Any at most countable group can be embedded as a subgroup in a group G with the following properties*:

1. (1) *G is an uncountable group*;
2. (2) *$G = \bigcup_{\alpha\in\omega_1} G_\alpha$, where each G_α is a countable subgroup of G_β for any $\beta \ni \alpha$, each G_α is malnormal in G, and all of the G_α* (*except possibly G_1*) *are topologizable*;
3. (3) *under CH, $G = S^{10000}$ for any uncountable $S \subset G$* (*this means that G is a Jonsson semigroup, i.e., all proper subsemigroups of G are countable*) *and G is nontopologizable*;
4. (4) *under CH, G is simple*;
5. (5) *under CH, $G \setminus \{1\}$ is unconditionally closed but not algebraic.*

Remark 2. Lemma 1 may be useful for constructing an example in ZFC. The nontopologizable group constructed by G. Hesse in ZFC in his dissertation [6] is very likely to have such a structure.

Appendix

We begin this section with mentioning some basic definitions and facts from [9]; see [9] for more details.

Definition A.1. Suppose that K and L are groups, $H \subset K$ and $H' \subset L$ are their isomorphic subgroups, and $\varphi\colon H \to H'$ is an isomorphism. The free product of K and L with the subgroups H and H' amalgamated by the isomorphism φ is the quotient of the free product $K * L$ by the relations $\varphi(h) = h$ for all $h \in H$. In what follows, we identify H with H' (i.e., assume that $K \cap L = H$) and refer to the free product of K and L with H and H' amalgamated by φ as the *free product of K and L with amalgamation over H* or simply the *amalgamated free product of K and L*. We use the standard notation $K *_H L$ for the amalgamated free product.

The groups K and L are naturally embedded in $K *_H L$ (see [9]).

We set $L^* = K *_H L$ and identify the groups K and L with their images in L^* under the natural embeddings. We refer to elements of L^* as *words* and to elements of K and L as *letters*.

A *normal form* of a nonidentity element $w \in L^*$ is a sequence $g_1 \dots g_n$ of letters such that $w = g_1 \dots g_n$ in L^*, g_i and g_{i+1} belong to different factors (K and L) for any $i = 1, \dots, n-1$, and if $n \ne 1$, then none of the letters $g_1, \dots, g_n$ belongs to H. Any element w of L^* can be written in normal form. Moreover, it may have many normal forms, but the

number of letters in each of its normal forms is the same (see [9]); it is called the *length* of w and denoted by $|w|$.

Lemma A.1. *Any two normal forms $x_1 \dots x_n$ and $y_1 \dots y_n$ of the same element of L^* are related as follows: there exist $h_1, \dots, h_{n-1} \in H$ such that $y_1 = x_1 h_1^{-1}$, $y_2 = h_1 x_2 h_2^{-1}$, $y_3 = h_2 x_3 h_3^{-1}$, $\dots$, $y_n = h_{n-1} x_n$.*

Proof. We have $y_n^{-1} \dots y_1^{-1} x_1 \dots x_n = 1$. The normal form theorem for amalgamated free products [9, Theorem IV.2.6] asserts that if $z_1 \dots z_n$ is a normal form of some word, then either $n = 1$ and $z_1 = 1$ or this word is not 1. Thus, $y_n^{-1} \dots y_1^{-1} x_1 \dots x_n = 1$ is not a normal form, i.e., the letters y_1^{-1} and x_1 belong to the same factor. For definiteness, we assume that $x_1, y_1^{-1} \in K$. Suppose that $y_1^{-1} x_1 \notin H$. Let $z = y_1^{-1} x_1$. Since the forms $x_1 \dots x_n$ and $y_1 \dots y_n$ are normal, it follows that $x_2, y_2^{-1} \in L \setminus H$. Therefore, $y_n^{-1} \dots y_2^{-1} z x_2 \dots x_n$ is a normal form, which contradicts its being equal to 1. Thus, $y_1^{-1} x_1 = h_1$ for some $h_1 \in H$, whence $y_1 = x_1 h_1^{-1}$. We set $y_2' = h_1^{-1} y_2$. Consider the word $y_n^{-1} \dots {y_2'}^{-1} x_2 \dots x_n$. It equals 1; therefore, it is not a normal form. Arguing as above, we conclude that ${y_2'}^{-1}$ and x_2 cancel each other, i.e., ${y_2'}^{-1} x_2 = h_2 \in H$, i.e., $y_2^{-1} h_1 x_2 = h_2$, whence $y_2 = h_1 x_2 h_2^{-1}$. Continuing, we obtain the required $h_1, \dots, h_n$. $\square$

A word w is said to be *cyclically reduced* if it has a normal form $g_1 \dots g_n$ such that $n \le 1$ or g_1 and g_n belong to different factors (Lemma A.1 implies that any normal form of a cyclically reduced word has this property). A word $w = g_1 \dots g_n$ in normal form is *weakly cyclically reduced* if $n \le 1$ or $g_n g_1 \notin H$.

Let u and v be words with normal forms $u = g_1 \dots g_n$ and $v = h_1 \dots h_m$. If $g_n h_1 \in H$, then we say that g_n and h_1 *cancel* each other in the product uv. If g_n and h_1 belong to the same factor but $g_n h_1 \notin H$, then we say that g_n and h_1 *merge* in the normal form of the product uv. A representation $u_1 \dots u_k$ (where the u_i are words) of a word w is *semireduced* if there are no cancellations in the product $u_1 \dots u_k$; mergings are allowed. If the product contains neither cancellations nor mergings, then the representation is said to be *reduced*.

A subset R of the group L^* is called *symmetrized* if $r \in R$ implies that r is weakly cyclically reduced and all weakly cyclically reduced conjugates of r and r^{-1} belong to R. The *symmetrized closure* of an element (or a set of elements) of L^* is the least symmetrized set containing this element (or set). A word b is called a *piece* (with respect to a symmetrized set R) if there exist different $r, r' \in R$ and some $c, c' \in L^*$ such that $r = bc$, $r = bc'$, and these representations are semireduced.

Let $\lambda > 0$.

We say that a symmetrized set R satisfies the *small cancellation condition $C'(\lambda)$* if it has the following property.

The condition $C'(\lambda)$. If $r \in R$ has a semireduced representation $r = bc$, where b is a piece, then $|b| < \lambda|r|$; moreover, $|r| > 1/\lambda$ for all $r \in R$.

Lemma A.2. *Suppose that x and y are good fellows in K over H, $a \in L \setminus H$, $a^{-1}Ha \cap H = \{1\}$, and $h \in K \setminus H$. Then the symmetrized closure R of the word*

$$r_0 = hayaxa(ya)^2xa(ya)^3 \dots xa(ya)^{80}$$

satisfies the condition $C'(\lambda)$.

Proof. Clearly, any weakly cyclically reduced element of the group L^* is conjugate to a cyclically reduced element by means of an element of $K \cup L$. By Theorem IV.2.8 from [9], any cyclically reduced element of R is conjugate to a cyclic permutation of $r_0^{\pm 1}$ by means of an element of H. Thus, any element of R is conjugate to a cyclic permutation of $r_0^{\pm 1}$ by means of an element of $K \cup L$ and hence has length 6640 ($= |r_0|$) or 6641.

Take two elements $r, r' \in R$. Let us show that if they have normal forms in which the initial fragments of length larger than 600 coincide, then these elements themselves coincide. Suppose that

$$r = z_0 z_1 \dots z_n \qquad \text{and} \qquad r' = z_0' z_1' \dots z_n'$$

are normal forms and $z_i = z_i'$ for $i = 0, 1, \dots, s$, where $s \geq 600$. We have

$$z_0 z_1 \dots z_n = t\tilde{z}_1 \dots \tilde{z}_n t^{-1} \qquad \text{and} \qquad z_0' z_1' \dots z_n' = t'\tilde{z}_1' \dots \tilde{z}_n' t'^{-1},$$

where $t, t' \in K \cup L$ and $\tilde{z}_1 \dots \tilde{z}_n$ and the words $\tilde{z}_1' \dots \tilde{z}_n'$ are cyclic permutations of r_0^{ε} and r_0^{δ} for some $\varepsilon, \delta = \pm 1$. For definiteness, suppose that $\delta = 1$. Clearly, we can assume that t and $\tilde{z}_1$ belong to different factors (otherwise, we replace t by $t\tilde{z}_1$ and consider the cyclic permutation $\tilde{z}_2 \dots \tilde{z}_n \tilde{z}_1$ of r_0^{ε}); similarly, we can assume that t and $\tilde{z}_1$ belong to different factors as well. Then $\tilde{z}_n$ and t^{-1} belong to the same factor, i.e., $\tilde{z}_n t^{-1} = u \in K \cup L$, and $t\tilde{z}_1 \dots \tilde{z}_{n-1} u$ is a normal form. Similarly, $t'\tilde{z}_1' \dots \tilde{z}_{n-1}' u'$ is a normal form for some $u \in K \cup L$. By Lemma A.1, there exist $\tilde{h}_0, \dots, \tilde{h}_s, \tilde{h}_0', \dots, \tilde{h}_s' \in H$ for which

$$\begin{aligned} t\tilde{h}_0^{-1} &= z_0 = z_0' = t'\tilde{h}_0'^{-1}, \\ \tilde{h}_0 \tilde{z}_1 \tilde{h}_1^{-1} &= z_1 = z_1' = \tilde{h}_0' \tilde{z}_1' \tilde{h}_1'^{-1}, \\ \tilde{h}_1 \tilde{z}_2 \tilde{h}_2^{-1} &= z_2 = z_2' = \tilde{h}_1' \tilde{z}_2' \tilde{h}_2'^{-1}, \\ &\dots, \\ \tilde{h}_{s-1} \tilde{z}_s \tilde{h}_s^{-1} &= z_s = z_s' = \tilde{h}_{s-1}' \tilde{z}_s' \tilde{h}_s'^{-1}. \end{aligned} \tag{1}$$

Hence there exist $h_0, \dots, h_s \in H$ such that

$$t' = th_0^{-1} \quad \text{and} \quad \tilde{z}_i' = h_{i-1} \tilde{z}_i h_i^{-1} \quad \text{for} \quad i \leq s.$$

Each of the letters z_i and z'_i is $x^{\pm 1}$, $y^{\pm 1}$, $a^{\pm 1}$, or $h^{\pm 1}$. Since x and y are good fellows over H and $x, y, h \in K \setminus H$, while $a \in L \setminus H$, it follows that (i) $\tilde{z}_i = a^\varepsilon \iff \tilde{z}'_i = a$; (ii) $\tilde{z}_i = x^\varepsilon$ or $\tilde{z}_i = h^\varepsilon \iff \tilde{z}'_i = x$ or $\tilde{z}'_i = h$; (iii) $\tilde{z}_i = y^\varepsilon$ or $\tilde{z}_i = h^\varepsilon \iff \tilde{z}'_i = y$ or $\tilde{z}'_i = h$.

Suppose that $\tilde{z}_i \neq \tilde{z}'_i$, i.e., (ii) or (iii) holds. For definiteness, we assume that $\tilde{z}_i = x^\varepsilon$ and $\tilde{z}'_i = h$. If $i \le s-8$, then $\tilde{z}'_{i+2} = y$, $\tilde{z}'_{i+4} = x$, $\tilde{z}'_{i+6} = y$, and $\tilde{z}'_{i+8} = y$, while, certainly, either $\tilde{z}_{i+2} = \tilde{z}_{i+4} = y^\varepsilon$ or $\tilde{z}_{i+2} = y^\varepsilon$, $\tilde{z}_{i+4} = h^\varepsilon$, $\tilde{z}_{i+6} = y^\varepsilon$, and $\tilde{z}_{i+8} = x^\varepsilon$. If $i > s-8$, then $\tilde{z}'_{i-2} = \tilde{z}'_{i-4} = \dots = \tilde{z}'_{i-160} = y$, while at least one of the corresponding letters $\tilde{z}_j$ is x^ε. In any case, there exists a $j \le s$ such that $\tilde{z}_j = x^\varepsilon$ and $\tilde{z}'_j = y$ or $\tilde{z}_j = y^\varepsilon$ and $\tilde{z}'_j = x$, which is impossible.

Thus, we have $\tilde{z}^\varepsilon_i = \tilde{z}'_i$ for any $i \le s$. Clearly, the word $\tilde{z}'_1 \dots \tilde{z}'_s$ (being a cyclic permutation of r_0) contains a fragment of the form $xa(ya)^k xa(ya)^{k+1}xa$. The corresponding fragment of the word $\tilde{z}_1 \dots \tilde{z}_s$ must have the form $x^\varepsilon a^\varepsilon (y^\varepsilon a^\varepsilon)^k x^\varepsilon a^\varepsilon (y^\varepsilon a^\varepsilon)^{k+1} x^\varepsilon a^\varepsilon$, which implies $\varepsilon = 1$. These fragments, together with their positions in the words $\tilde{z}_1 \dots \tilde{z}_s$ and $\tilde{z}'_1 \dots \tilde{z}'_s$ (which are initial fragments of cyclic permutations of r_0), uniquely determine the permutations. We conclude that $\tilde{z}_1 \dots \tilde{z}_s$ coincides with $\tilde{z}'_1 \dots \tilde{z}'_s$. It remains to show that $t = t'$.

As mentioned above (see (1), there exist $\tilde{h}_0, \tilde{h}_1, \tilde{h}_2, \tilde{h}'_0, \tilde{h}'_1, \tilde{h}'_2 \in H$ such that

$$t\tilde{h}_0^{-1} = t'\tilde{h}'^{-1}_0,$$

$$\tilde{h}_0 \tilde{z}_1 \tilde{h}_1^{-1} = \tilde{h}'_0 \tilde{z}'_1 \tilde{h}'^{-1}_1 \quad (\text{i.e., } \tilde{z}_1^{-1} \tilde{h}_0^{-1} \tilde{h}'_0 \tilde{z}'_1 \in H),$$

and

$$\tilde{h}_1 \tilde{z}_2 \tilde{h}_2^{-1} = \tilde{h}'_1 \tilde{z}'_2 \tilde{h}'^{-1}_2 \quad (\text{i.e., } \tilde{z}_2^{-1} \tilde{h}_1^{-1} \tilde{h}'_1 \tilde{z}'_2 \in H).$$

One of the letters $\tilde{z}_1 = \tilde{z}'_1$ and $\tilde{z}_2 = \tilde{z}'_2$ is a. If $\tilde{z}_1 = \tilde{z}'_1 = a$, then $h_0 = h'_0$ (because $a^{-1}Ha \cap H = \{1\}$ by assumption), whence $t = t'$; if $\tilde{z}_2 = \tilde{z}'_2 = a$, then $h_1 = h'_1$, whence $h_0 = h'_0$ (because $\tilde{z}_1 = \tilde{z}'_1$) and $t = t'$.

Let b be a piece. This means by definition such that b has two normal forms coinciding (up to their last letters) with initial fragments of normal forms of two different element r and r' in R; i.e., that there are different normal forms $z_0 z_1 \dots z_n$ and $z'_0 z'_1 \dots z'_n$ in R such that $b = z_0 z_1 \dots z_s u = z'_0 z'_1 \dots z'_s u'$, where $s < n$ and u and u' are some (possibly identity) letters. We have shown that $s < 600$ (otherwise, the forms $z_0 z_1 \dots z_n$ and $z'_0 z'_1 \dots z'_n$ would coincide). It follows that $|b| \le 601 < \frac{1}{10}6640$. It remains to recall that all elements of R have length 6640 or 6641. □

Theorem V.11.2 from [9] asserts, in particular, that if N is the normal closure of a symmetrized set R in $L^* = K *_H L$ and R satisfies the condition $C'(1/10)$, then the natural homomorphism $L^* \to L^*/N$ acts as an endomorphism on K and L; moreover, any nonidentity element

w of N has a reduced representation $w = usv$, where $|s| > \frac{7}{10}|r|$ for some $r \in R$ (and hence $|w| > 7$) and r has a reduced representation of the form $r = st$.

Let $\varphi\colon L^* \to L^*/N$ be the natural homomorphism.

Lemma A.3. *If the conditions of Lemma* A.2 *hold and H is malnormal in L, then $\varphi(K)$ is malnormal in L^*/N.*

Proof. Suppose that $\varphi(K)$ is not malnormal in L^*/N. Take $u \in L^*$ such that $\varphi(u) \in L^*/N \setminus \varphi(K)$ (i.e., $u \notin KN$) and $\varphi(u)^{-1}\varphi(g)\varphi(u) = \varphi(g')$ for some $g, g' \in K \setminus \{1\}$. This means that $u^{-1}gug'^{-1} \in N$ for some $g, g' \in K \setminus \{1\}$, or, equivalently, $gu^{-1}g'ug'' \in N$ for some $g, g', g'' \in K$ such that $g' \neq 1$ and $gg'' \neq 1$. Suppose that u is a shortest word from $L^* \setminus KN$ for which such g, g', and g'' exist. Let $u_1 \dots u_n$ be a normal form of u. If $u_n \in K$, then $gu_n^{-1} = gg''g''^{-1}u_n^{-1} \in K$ and $u_ng'' = u_ng^{-1}gg'' \in K$; replacing g'' by u_ng'' and g by gu_n^{-1}, we see that $u_1 \dots u_{n-1}$ is a word with the same properties as u but shorter than u. Thus, $u_n \notin K$, i.e., $u_n \in L \setminus H$.

If $u_1^{-1}g'u_1 = 1$, then $gu^{-1}g'ug'' = gg''$. As mentioned above, any nonidentity element of N has length at least 7; hence $gg'' = 1$, which contradicts the assumption. Therefore, $u_1^{-1}g'u_1 \neq 1$. If $u_1 \in K$, then, replacing g' by $u_1^{-1}g'u_1$, we see that $u_2 \dots u_n$ is a word with the same properties as u but shorter than u. Thus, $u_1 \notin K$, i.e., $u_1 \in L \setminus H$.

If u has a reduced representation vsw, where s is a fragment of some $r \in R$ (i.e., r has a reduced representation $r = s_1ss_2$), then $\varphi(u) = \varphi(vs_1^{-1}s_2^{-1}w)$, because $vs_1^{-1}s_2^{-1}s^{-1}v^{-1} \in N$ (the element $s_1^{-1}s_2^{-1}s^{-1}$ is a cyclic permutation of $r^{-1} = s_2^{-1}s^{-1}s_1^{-1}$ and hence belongs to R). Thus, we have $|s| \leq |s_1| + |s_2|$ (otherwise, the word u is not shortest); i.e., u cannot contain a fragment of a word $r \in R$ of length $> \frac{1}{2}|r|$.

Let us find a normal form of $gu^{-1}g'ug''$. If $g, g', g'' \notin H$ (i.e., $g, g', g'' \in K \setminus H$), then $gu_n^{-1} \dots u_1^{-1}g'u_1 \dots u_ng''$ is a normal form, because, as shown above, $u_1, u_n \in L \setminus H$. If $g \in H$ and $g', g'' \notin H$, then $u_n'^{-1} \dots u_1^{-1}g'u_1 \dots u_ng''$, where $u_n'^{-1} = gu_n^{-1}$, is a normal form (clearly, $gu_n^{-1} \in L \setminus H$). If $g' \in H$ and $g, g'' \notin H$, then $gu_n^{-1} \dots u_2^{-1}u_0u_2 \dots u_ng''$, where $u_0 = u_1^{-1}g'u_1$, is a normal form. (Indeed, we have $u_1 \in L \setminus H$ and $g' \in H \setminus \{1\}$; since H is malnormal in L, it follows that $u_1^{-1}g'u_1 \in L \setminus H$.) The remaining cases are considered similarly.

Thus, in any case, $gu^{-1}g'ug''$ has a normal form equal (up to the first and last letters) to $u_n^{-1} \dots u_2^{-1}\tilde{u}u_2 \dots u_n$, where $\tilde{u}$ is the word $u_1^{-1}g'u_1$ or the letter from $L \setminus H$ equal to $u_1^{-1}g'u_1$.

As mentioned above, any nonidentity element of N is a reduced product of a fragment s of some word $r \in R$ of length $> \frac{7}{10}|r|$ and something else. Every $r \in R$ is a cyclic permutation of r_0 conjugate by means of some letter. Thus, the normal form of $gu^{-1}g'ug''$ contains a long (of length $> \frac{7}{10}|r_0| - 2$) fragment t of a cyclic permutation of r_0. Since u can contain only fragments of length $\leq \frac{1}{2}|r|$, it follows

that $t = u_k^{-1} \dots u_2^{-1}\tilde{u}u_2 \dots u_m$, where $k, m > \frac{1}{10}|r|$. Let the fragment t be $z_1 \dots z_l$, where $l = k + m + 1$ or $l = k + m + 3$ (depending on $\tilde{u}$). According to Lemma A.1, for each $i \le l$, the ith letter in $u_k^{-1} \dots u_2^{-1}\tilde{u}u_2 \dots u_m$ belongs to Hz_iH. Since k and m are large and $\tilde{u}$ contains one or three letters, there exists a $j \in \{2, ..., \min\{k, m\}\}$ such that $u_j^{-1} \in Hx^{\pm 1}H$ and $u_j \in Hy^{\pm 1}H$ (x and y are the same as in Lemma A.2). This contradicts the x and y being good fellows over H. □

We identify K with $\varphi(K)$ and L with $\varphi(L)$, that is, treat K and L as subgroups of $(L *_H K)/N$.

The following fact was kindly communicated to the author by Anton Klyachko.

Lemma A.4. *Suppose that L and K are infinite countable groups, $L \cap K = H$, $x, y \in K$ are good fellows over H, $a \in L$, $a^{-1}Ha \cap H = \{1\}$ in L, $h \in K \setminus H$, and*

$$r_0 = hayaxa(ya)^2xa(ya)^3 \dots xa(ya)^{80} \in L *_H K.$$

Let R be the symmetrized closure of r_0, and let N be the normal closure of R. Then the group $\langle L *_H K \mid r_0 = 1\rangle = (L *_H K)/N$ admits a nondiscrete Hausdorff group topology.

Proof. Let us enumerate the elements of $L *_H K$:

$$L *_H K = \{1, g_1, g_2, \dots\}.$$

We shall construct nontrivial normal subgroups N_1, N_2, … of $L *_H K$ such that $N_{i+1} \subset N_i$ and $g_i \notin N_i$ for each i.

Take cyclically reduced words r_n in $L *_H K$ such that their lengths unboundedly increase and the symmetrized closure of $\{r_n : n \ge 0\}$ (and, therefore, of any subset of this set) satisfies $C'(1/10)$; in particular, each word in the normal subgroup generated by the (symmetrized closure of) $\{r_n : n \ge k\}$ is at least half as long as r_k. For such words we can take

$$r_n = xa(ya)^{80(n-1)+1}xa(ya)^{80(n-1)+2} \dots xa(ya)^{80n}.$$

This is proved in precisely the same way as Lemma A.2. The only difference is that $\tilde{z}_1 \dots \tilde{z}_n$ and $\tilde{z}'_1 \dots \tilde{z}'_n$ may be cyclic permutations of r_k^ε and $r_{k'}^\delta$ for different k and k'. This does not matter, because if, say, $|r_k^\varepsilon| \le |r_{k'}^\delta|$ and $|s| > \frac{1}{10}|r_k^\varepsilon|$, then the word $\tilde{z}_1 \dots \tilde{z}_s$ (as well as $\tilde{z}'_1 \dots \tilde{z}'_s$), being a cyclic permutation of r_k^ε, still contains a fragment of the form

$$xa(ya)^jxa(ya)^{j+1}xa \quad \text{or} \quad a^{-1}x^{-1}(a^{-1}y^{-1})^{j+1}a^{-1}x^{-1}(a^{-1}y^{-1})^ja^{-1}x^{-1},$$

which determines k, ε, and the permutation.

For every $n \in \omega$, let $k(n)$ be an integer such that the word $r_{k(n)}$ is twice as long as g_n; we assume that $k(n+1) > k(n)$. We define N_n to be the normal subgroup generated by $\{r_k : k \ge k(n)\}$. It does not contain g_n, because g_n is too short. Therefore, $\bigcap N_n = \{1\}$.

On the other hand, $N_n \not\subset NN_{n+1}$ for any n; for example, $r_{k(n)} \notin NN_{n+1}$ for any $n > 0$. Indeed, NN_{n+1} is the normal closure of the set $\{r_0\} \cup \{r_k : k \geq k(n+1)\}$ and, therefore, of the symmetrized closure R_{n+1} of this set, which satisfies the condition $C'(1/10)$. By above-cited Theorem V.11.2 from [9], each element of NN_{n+1} must contain a fragment s of some $r \in R_{n+1}$ of length at least $\frac{7}{10}|r|$, while $r_{k(n)}$ cannot contain such a fragment. Indeed, if $r_{k(n)} = usv$ is a reduced representation and s is a long fragment of r, i.e., r has a reduced representation $u'sv'$, then svu (which is a cyclic permutation of $r_{k(n)}$) is a reduced representation of some word $\tilde{r}$ from the symmetrized closure of $r_{k(n)}$, and $sv'u'$ (which is a cyclic permutation of a weakly cyclically reduced conjugate of some word in $\{r_0\} \cup \{r_k : k \geq k(n+1)\}$) is a semireduced representation of some word $\tilde{r}'$ from the symmetrized closure R_{n+1} of $\{r_0\} \cup \{r_k : k \geq k(n+1)\}$. Thus, s is a piece with respect to the symmetrized closure of $\{r_n : n \geq 0\}$ (which satisfies the small cancellation condition $C'(1/10)$), and it cannot be longer than $\frac{1}{10}|\tilde{r}'|$. Clearly, $|\tilde{r}'| \leq |r| + 1$, and $|s| \leq \frac{1}{10}|\tilde{r}'| < \frac{7}{10}|r|$, which contradicts the choice of s.

Thus, the images of the groups N_n under the natural homomorphism $L *_H K \to (L *_H K)/N$ form a strictly decreasing sequence of nontrivial normal subgroups with trivial intersection. Clearly, such subgroups constitute a neighborhood base at the identity for some nondiscrete Hausdorff group topology on $(L *_H K)/N$. □

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Department of Mechanics and Mathematics, Moscow State University, Leninskie Gory, Moscow, 119992 Russia

E-mail address: o-sipa@yandex.ru